Lecture 9: Systems

Introduction

Systems

An important component of signal processing is a system that usually abstracts a physical process. A system is characterized by its ability to accept a set of input signals $x_i(t)$ and to produce a set of output signals $y_j(t)$,in response to the input signals. In other words, a system establishes a relationship between a set of inputs and the corresponding set of outputs.

Systems

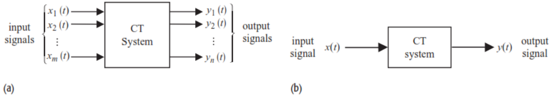

Fig. 2.1. General schematics of CT systems. (a) Multiple-input, multiple-output (MIMO) CT system with m inputs and n outputs. (b) Single-input, single-output CT system.

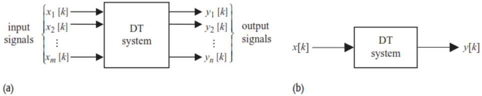

Figure 1: System types

Systems

$$\begin{aligned} \text{CT system} \quad & x(t) \to y(t) \\ \text{DT system} \quad & x[k] \to y[k] \end{aligned} \tag{1}$$

RLC Circuit

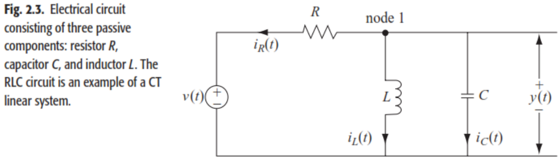

Figure 2: RLC circuit

RLC Circuit

We apply Kirchhoff's current law to node 1, shown in the top branch of the RLC circuit in Fig. 2 The equations for the currents flowing out of node 1 along resistor R, inductor L, and capacitor C, are given by

$$
\begin{aligned}
&\text{resistor } R \qquad && i_R = \frac{y(t) - \mu(t)}{R} \\
&\text{inductor } L \qquad && i_L = \frac{1}{L}\int_{-\infty}^{t} y(\tau)d\tau \\
&\text{capacitor } C \qquad && i_C = C\frac{dy}{dt}
\end{aligned} \tag{2}
$$

RLC Circuit

Total current in node 1 iz zero

$$\frac{y(t)-\mu(t)}{R}+\frac{1}{L}\int_{-\infty}^{t} y(\tau)d\tau + C\frac{dy}{dt}=0 \tag{3}$$

After differentiating and arranging terms we get

$$\frac{d^2y}{dt^2}+\frac{1}{RC}\frac{dy}{dt}+\frac{1}{LC}y(t)=\frac{1}{RC}\frac{d\mu}{dt} \tag{4}$$

Amplitude modulator

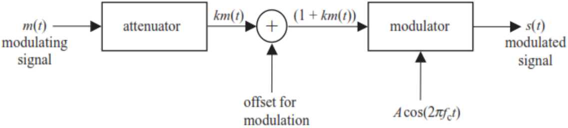

Figure 3: Amplitude modulation

$$s(t) = A(1 + km(t)) \cos(2\pi f_c t) \quad (5)$$

Classification of systems

(i) linear and non-linear systems;

(ii) time-invariant and time-varying systems;

(iii) systems with and without memory;

(iv) causal and non-causal systems;

(v) invertible and non-invertible systems;

(vi) stable and unstable systems.

Linear and non-linear systems

A CT system with the following set of inputs and outputs:

$$x_1(t) \to y_1(t) \quad \text{and} x_2(t) \to y_2(t)$$

System is linear if it satisfies

$$\begin{aligned} &\text{additive property} && x_1(t) + x_2(t) \to y_1(t) + y_2(t) \\ &\text{homogeneity property} && \alpha x_1(t) \to \alpha y_1(t) \end{aligned} \tag{6}$$

above equations can be generalized in one

$$\alpha x_1(t) + \beta x_2(t) \to \alpha y_1(t) + \beta y_2(t) \tag{7}$$

For a DT system

$$x_1[k] \to y_1[k] \quad \text{and} x_2[k] \to y_2[k]$$

We have

$$\alpha x_1[k] + \beta x_2[k] \to \alpha y_1[k] + \beta y_2[k] \tag{8}$$

Special case of linearity property is *zero-input zero-output property*.

Differentiator

$$y(t) = \frac{dx}{dt}$$

Let take two inputs and two corresponding outputs and let see if system is linear or not

$$x_1(t) \to \frac{dx_1(t)}{dt} = y_1(t) \qquad (9)$$

and

$$x_2(t) \to \frac{dx_2(t)}{dt} = y_2(t) \qquad (10)$$

We get

$$\alpha x_1(t)+\beta x_2(t) \to \frac{d}{dt}(\alpha x_1(t)+\beta x_2(t)) = \alpha\frac{dx_1(t)}{dt}+\beta\frac{dx_2(t)}{dt} \qquad (11)$$

but

$$\alpha\frac{dx_1(t)}{dt} + \beta\frac{dx_2(t)}{dt} = \alpha y_1(t) + \beta y_2(t) \qquad (12)$$

This means that differentiator is a linear system.

Amplifier with additive bias

$$y(t) = 3x(t) + 5 \tag{13}$$

we can write

$$x_1(t) \to 3x_1(t) + 5 = y_1(t) \tag{14}$$

and

$$x_2(t) \to 3x_2(t) + 5 = y_2(t) \tag{15}$$

$\alpha x_1(t) + \beta x_2(t)$ will be processed by system as following

$$\alpha x_1(t) + \beta x_2(t) \to 3(\alpha x_1(t) + \beta x_2(t)) + 5 \tag{16}$$

but

$$3(\alpha x_1(t) + \beta x_2(t)) + 5 = \alpha y_1(t) + \beta y_2(t) - 5 \tag{17}$$

So amplifier with additive bias is not a linear system.
One of the way to check system is linear or not is to check zero input property. If zero input gives non zero output system is not linear.

Incrementally linear system

In the last example we had.

$$\begin{aligned} \text{intput } x_1(t) \quad y_1(t) = 3x_1(t) + 5 \\ \text{intput } x_2(t) \quad y_2(t) = 3x_2(t) + 5 \end{aligned} \tag{18}$$

Calculating the difference on both sides of the above equations yield

$$\begin{gathered} y_2(t) - y_1(t) = 3(x_2(t) - x_1(t)) \\ \Delta y(t) = 3\Delta x(t) \end{gathered} \tag{19}$$

this means that change in the output of system is linearly related to the change in the input. This kind of systems are called incrementally linear systems.

Incrementally linear system

An incrementally linear system can be expressed as a combination of a linear system and an adder that adds an offset $y_{zi}(t)$ to the output of the linear system. The value of offset $y_{zi}(t)$ is zero-input response of original system. System S_1, $y(t) = 3x(t) + 5$, for example, can be expressed as a combination of a linear system S_2 , $y(t) = 3x(t)$, plus an offset given by the zero-input response of S_1 , which equals $y_{zi}(t) = 5$.

Fig. 2.10. Incrementally linear system expressed as a linear system with an additive offset.

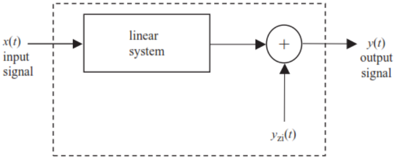

Figure 4: Block diagram representation of an incrementally linear system in terms of a linear system and an additive offset y_{zi}

Numerical differentiation and integration

Let see how differentiation and integration work in DT case. We will see, that CT differentiator and integrator lead to finite difference equations that are frequently used to describe DT systems. In numerical methods space and time are sampled(discretized) by some value ofΔx and Δt. Let consider time derivative at $t = k\Delta t$ time moment. There are several methods to calculate time derivative. We use forward difference method.

$$\frac{dx}{dt}\Big|_{t=k\Delta t} \approx \frac{x(k\Delta t) - x((k-1)\Delta t)}{\Delta t} \tag{20}$$

we get

$$y(k\Delta t) = \frac{x(k\Delta t) - x((k-1)\Delta t)}{\Delta t} \tag{21}$$

or

$$y(k\Delta t) = C_1(x(k\Delta t) - x((k-1)\Delta t)) \tag{22}$$

where $C_1 = \dfrac{1}{\Delta t}$

Numerical differentiation and integration

Usually, the sampling interval Δt in the last equation above is omitted, resulting in the following expression:

$$y[k] = C_1(x[k] - x[k-1]) \tag{23}$$

which is a finite-difference representation of the differentiator. shown in Figure 5

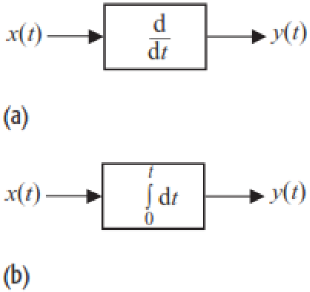

Figure 5: Differenciator (a) and Integrator (b)

Numerical differentiation and integration

Now consider numerical integration

$$\int_{(k-1)\Delta t}^{k\Delta t} x(t)dt \approx x((k-1)\Delta t) \cdot \Delta t \tag{24}$$

This integral above is area under function $x(t)$ for time interval $[(k-1)\Delta t, k\Delta t]$
Calculating integral at time moment $t = k\Delta t$ we have

$$y(t)|_{t=k\Delta t} = \int_0^t x(t)dt = \int_0^{(k-1)\Delta t} x(t)dt + \int_{(k-1)\Delta t}^{k\Delta t} x(t)dt \tag{25}$$

The first term is $y((k-1)\Delta t)$ because of definition of $y(t)$. The second term we have calculated above.

Numerical differentiation and integration

So we have

$$y(t)(k\Delta t) = y((k-a)\Delta t) + \Delta t x((k-1)\Delta t) \tag{26}$$

Omitting Δt we have

$$y[k] = y[k-1] + C_2 x[k-1] \tag{27}$$

where $C_2 = \Delta t$