Lecture 11: Time-domain analysis of LTIC systems

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System examples

Relation

$$x(t) \to y(t) \tag{1}$$

We can describe as following. Denote by $\hat{\boldsymbol{T}}$ operation which system does on input $x(t)$. So instead of relation notation above we can use notation

$$\hat{\boldsymbol{T}}\{x(t)\} = y(t) \tag{2}$$

In this notations linearity property means

$$\hat{\boldsymbol{T}}\{\alpha x_1(t) + \beta x_2\} = \alpha\hat{\boldsymbol{T}}\{x_1(t)\} + \beta\hat{\boldsymbol{T}}\{x_2(t)\} \tag{3}$$

System example 1

Show that system described with equation

$$y(t) = tx(t) \tag{4}$$

is linear.

Solution: $y_1(t) = \hat{\boldsymbol{T}}x_1 = tx_1(t)$ and $y_2(t) = \hat{\boldsymbol{T}}x_2 = tx_2(t)$. Let take combination of two inputs x_1 and x_2

$$x_3(t) = \alpha x_1 + \beta x_2 \tag{5}$$

Now let check what System does with x_3

$$\hat{\boldsymbol{T}}\{x_3(t)\} = \hat{\boldsymbol{T}}\{\alpha x_1 + \beta x_2\} = t(\alpha x_1 + \beta x_2) = \alpha t x_1 + \beta t x_2 \tag{6}$$

but $tx_1(t) = y_1(t)$ and $tx_2(t) = y_2(t)$ So we have

$$\hat{\boldsymbol{T}}\{\alpha x_1 + \beta x_2\} = \alpha y_1 + \beta y_2 = \alpha\hat{\boldsymbol{T}}\{x_1(t)\} + \beta\hat{\boldsymbol{T}}\{x_2(t)\} \tag{7}$$

So system is linear.

System example 2

Show that system described with equation

$$y(t) = sin(x(t)) \tag{8}$$

is not linear.

Solution: $y_1(t) = \hat{\boldsymbol{T}}\{x_1\} = sin(x_1(t))$ and $y_2(t) = \hat{\boldsymbol{T}}\{x_2\} = sin(x_2(t))$. Let take combination of two inputs x_1 and x_2

$$x_3(t) = \alpha x_1 + \beta x_2 \tag{9}$$

Now let check what System does with x_3

$$\hat{\boldsymbol{T}}\{x_3(t)\} = \hat{\boldsymbol{T}}\{\alpha x_1 + \beta x_2\} = sin(\alpha x_1 + \beta x_2) \tag{10}$$

But $\alpha\hat{\boldsymbol{T}}\{x_1\} + \beta\hat{\boldsymbol{T}}\{x_2\} = \alpha sin(x_1(t)) + \beta sin(x_2(t))$ and this is not equal what we got above. So system is not linear.

Representation of LTIC systems

For a linear CT system, the relationship between the applied input $x(t)$ and output $y(t)$ can be described using a linear differential equation of the following form:

$$\begin{aligned}&\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y(t) \\ &= b_m\frac{d^m x}{dt^m} + b_{n-1}\frac{d^{m-1} x}{dt^{m-1}} + \cdots + b_1\frac{dx}{dt} + b_0 x(t)\end{aligned} \tag{11}$$

Example: RLC circuit

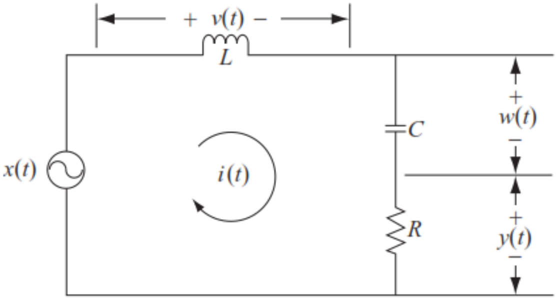

Figure 1: RLC circuit

Determine the input–output representations of the series RLC circuit shown in Fig. 1 for the three outputs $v(t)$, $w(t)$, and $y(t)$.

Example: RLC circuit

Solution:

Figure 1 illustrates an electrical circuit consisting of three passive components: resistor R, inductor L, and capacitor C. Applying Kirchhoff's voltage law, the relationship between the input voltage $x(t)$ and the loop current $i(t)$ is given by

$$x(t) = L\frac{di}{dt} + Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(t)dt \tag{12}$$

Differentiating with respect to t yields

$$L\frac{d^2i}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = \frac{dx(t)}{dt} \tag{13}$$

We consider three different outputs of the RLC circuit in the following discussion, and for each output we derive the differential equation modeling the input–output relationship of the LTIC system.

Relationship between x(t) and v(t)

Relationship between $x(t)$ and $v(t)$ The output voltage $v(t)$ is measured across inductor L. Expressed in terms of the loop current $i(t)$, the voltage $v(t)$ is given by

$$v(t) = L\frac{di}{dt} \tag{14}$$

Integrating the above equation with respect to t yields $i(t) = \frac{1}{L}\int v(t)dt$. By substituting the value of $i(t)$ into Eq. 13, we obtain

$$\frac{dv}{dt} + \frac{R}{L}v(t) + \frac{1}{LC}\int v(t)dt = \frac{dx(t)}{dt} \tag{15}$$

Relationship between x(t) and v(t)

The above input–output relationship includes both differentiation and integration operations. The integral operator can be eliminated by calculating the derivative of both sides of the equation with respect to t . This results in the following equation:

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv(t)}{dt} + \frac{1}{LC}v(t) = \frac{d^2x(t)}{dt^2} \qquad (16)$$

It can be shown that an LTIC system can always be modeled by a linear, constant coefficient differential equation with the appropriate initial conditions.

Relationship between x(t) and w(t)

The output voltage $w(t)$, measured across capacitor C, is given by

$$w(t) = \frac{1}{C} \int_{-\infty}^{t} i(t) dt \tag{17}$$

or

$$i(t) = C \frac{w(t)}{dt} \tag{18}$$

Substituting the value of $i(t)$ into Eq. (13) yields

$$LC \frac{d^2 w}{dt^2} + RC \frac{dw(t)}{dt} + w(t) = x(t) \tag{19}$$

This is also a linear, second-order, constant-coefficient differential equation.

Relationship between x(t) and y(t)

Finally, we measure the output voltage $y(t)$ across resistor R. Using Ohm's law, the output voltage $y(t)$ is given by $y(t) = i(t)R$. Substituting the value of $i(t) = y(t)/R$ into Eq. (13) yields

$$\frac{L}{R}\frac{d^2y}{dt^2} + \frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{dx(t)}{dt} \tag{20}$$

Compact form

More compact form of Eq. (23) is obtained using operator formalism. Namely, by denoting the differentiation operator d/dt by $\hat{\boldsymbol{D}}$:

$$\hat{\boldsymbol{D}}^n y + a_{n-1}\hat{\boldsymbol{D}}^{n-1} y + \cdots + a_1 \hat{\boldsymbol{D}} y + a_0 y(t) \\ = b_m \hat{\boldsymbol{D}}^m x + b_{n-1}\hat{\boldsymbol{D}}^{m-1} x + \cdots + b_1 \hat{\boldsymbol{D}} x + b_0 x(t) \tag{21}$$

or

$$\boldsymbol{Q}(\hat{\boldsymbol{D}}) y(t) = \boldsymbol{P}(\hat{\boldsymbol{D}}) x(t) \tag{22}$$

where $\boldsymbol{Q}(\hat{\boldsymbol{D}})$ is the nth-order differential operator, $\boldsymbol{P}(\hat{\boldsymbol{D}})$ is the mth-order differential operator, and the a_i and b_i are constants. Equation above is used extensively to describe an LTIC system.

Solving differential equations for LTIC systems

As we mentioned for a linear CT system, the relationship between the applied input $x(t)$ and output $y(t)$ is described using a linear differential equation of the following form:

$$\begin{aligned} &\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y(t) \\ &= b_m\frac{d^m x}{dt^m} + b_{n-1}\frac{d^{m-1}x}{dt^{m-1}} + \cdots + b_1\frac{dx}{dt} + b_0 x(t) \end{aligned} \tag{23}$$

where the a_k s and b_k s are constants, and the derivatives of the output signal $y(t)$ are known at a given time instant, say$t = t_0$. As usual time moment $t_0 = 0$ and values of $y(t)$ and its derivatives are called *initial conditions*.
The highest derivative in Eq. (1) denotes the order of the differential equation. Equation (1) is therefore either of order n or m.

Solving differential equations for LTIC systems

We solve eq. (1) in time domain without using any transform of variables. This kind of method is called a direct method.

Output $y(t)$ is sum of two components:

(i) zero-input response $y_{zi}(t)$. This component is associated with initial conditions.

(ii) zero-state response$y_{zs}(t)$ associated with the applied input$x(t)$.

$$y(t) = y_{zi}(t) + y_{zs}(t)$$

$y_{zi}(t)$ & $y_{zs}(t)$

The zero-input response $y_{zi}(t)$ is the component of the output $y(t)$ of the system when the input is set to zero. The zero-input response describes the manner in which the system dissipates any energy or memory of the past as specified by the initial conditions.
The zero-state response $y_{zs}(t)$ is the component of the output $y(t)$ of the system with initial conditions set to zero. It describes the behavior of the system forced by the input. In the following, we outline the procedure to evaluate the zero-input and zero-state responses.

Zero-input response $y_{zi}(t)$

The zero-input response $y_{zi}(t)$ is the output of the system when the input is zero. Hence, $y_{zi}(t)$ is the solution to the following homogeneous differential equation:

$$\sum_{k=0}^{n} a_k \frac{d^k y(t)}{dt^k} = 0 \tag{24}$$

with known initial conditions

$$y(t), \frac{dy}{dt}, \frac{d^2y}{dt^2}, \cdots \frac{d^{n-1}y(t)}{dt^{n-1}} \text{ at } t = 0 \tag{25}$$

To determine the zero-input response$y_{zi}(t)$, assume that the zero-input response is given by $y_{zi}(t) = Ae^{st}$, substitute $y_{zi}(t)$ in the homogeneous differential equation, Eq.(24), and solve the resulting equation.

Example: Calculate $y_{zi}(t)$

Let assume CT system is described by following DE(Differentian Equation)

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y(t) = 3x(t) \tag{26}$$

Compute the zero-input response of the system for initial conditions $y(0) = 2$ and $\dot{y}(0) = -5$

Solution:

Homogeneous equation is:

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y(t) = 0 \tag{27}$$

Substituting here $y_{zi}(t) = Ae^{st}$ yields

$$Ae^{st}(s^2 + 5s + 4) = 0 \tag{28}$$

Polynomial $P = s^2 + 5s + 4$ is called characteristic polynomial.

Example: Calculate $y_{zi}(t)$

For a finite t from eq.(28) we get (this kind of equation is called characteristic equation)

$$s^2 + 5s + 4 = 0 \tag{29}$$

We will to find root for more complex polynomials. You can use Matlab or Octave command window

```
>> s=[1,5,4];
>> roots(s)
ans =

-4
-1

>>
```

This equation has two roots $s = -1$ and $s = -4$.

Example: Calculate $y_{zi}(t)$

The zero-input solution is given by

$$y_{zi}(t) = A_0 e^{-t} + A_1 e^{-4t} \tag{30}$$

A_0 and A_1 ar constants and we can calculate those numbers using initial conditions

$$A_0 + A_1 = 2 \tag{31}$$
$$-A_0 - 4A_1 = -5$$

Solution of above equation is $A_0 = 1$ and $A_1 = 1$ and zero-input response is

$$y_{zi}(t) = e^{-t} + e^{-4t} \tag{32}$$

$y_{zi}(t)$ for repeated roots

When characteristic equation has repeated roots solution is following. Let assume root $s = a$ is repeated J times. We need to include J distinct terms in the zero-input response associated with a by using following J functions:

$$e^{at}, te^{at}, t^2e^{at}, \cdots, t^{J-1}e^{at} \tag{33}$$

The zero-input response of an LTIC system is then given by

$$y_{zi}(t) = A_0e^{at} + A_1te^{at} + A_2t^2e^{at} + \cdots + A_{J-1}t^{J-1}e^{at} \tag{34}$$

If there are two or more roots like a_1 and a_2 we need to include terms to similar manner as above for each repeated roots.