Lecture 13: Solving DE for LTIC

Introduction

Solving DE for LTIC

Now let go back to LTIC system. We got

$$\hat{\boldsymbol{D}}^n y + a_{n-1}\hat{\boldsymbol{D}}^{n-1} y + \cdots + a_1 \hat{\boldsymbol{D}} y dt + a_0 y(t)$$
$$= b_m \hat{\boldsymbol{D}}^m x + b_{n-1}\hat{\boldsymbol{D}}^{m-1} x + \cdots + b_1 \hat{\boldsymbol{D}} x + b_0 x(t) \tag{1}$$

or

$$\boldsymbol{Q}(\hat{\boldsymbol{D}}) y(t) = \boldsymbol{P}(\hat{\boldsymbol{D}}) x(t) \tag{2}$$

where $\boldsymbol{Q}(\hat{\boldsymbol{D}})$ is the nth-order differential operator, $\boldsymbol{P}(\hat{\boldsymbol{D}})$ is the mth-order differen- tial operator, and the a_i and b_i are constants. We know that solution of above differential equation is

$$y(t) = y_{zi}(t) + y_{zs}(t) \tag{3}$$

$y_{zi}(t)$ is the zero-input response of the system and $y_{zs}(t)$ is the zero-state response of the system.

Solving DE for LTIC

Note that the zero-input component $y_{zi}(t)$ is the response produced by the system because of the initial conditions (and not due to any external input), and hence $y_{zi}(t)$ is also known as the natural response of the system.

For example, the initial conditions may include charges stored in a capacitor or energy stored in a mechanical spring.

The zero-input response $y_{zi}(t)$ is evaluated by solving a homogeneous equation obtained by setting the input signal $x(t) = 0$. For Eq.(2), the homogeneous equation is given by

$$\boldsymbol{Q}(\hat{\boldsymbol{D}})y(t) = 0 \tag{4}$$

The zero-state response $y_{zs}(t)$ arises due to the input signal and does not depend on the initial conditions of the system. In calculation of the zero-state response, the initial conditions of the system are assumed to be zero.

Solving DE for LTIC

The zero-state response is also referred to as the forced response of the system since the zero- state response is forced by the input signal. For most stable LTIC systems, the zero-input response decays to zero as $t \to \infty$ since the energy stored in the system decays over time and eventually becomes zero. The zero-state response, therefore, defines the steady state value of the output.

LTIC Example

We go back to example of RLC circuit (See Lecture 11).

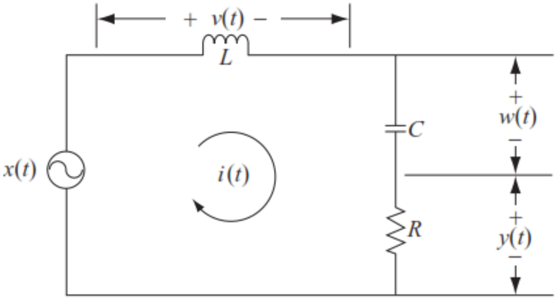

Figure 1: RLC circuit

Assume that the inductance $L = 0H$ (i.e. the inductor does not exist in the circuit), resistance $R = 5\Omega$, and capacitance $C = 1/20F$. Determine the output signal $y(t)$ when the input voltage is given by $x(t) = \sin(2t)$ and the initial voltage $y(0^-) = 2V$ across the resistor.

LTIC Example

Solution:

We got DE for $y(t)$

$$\frac{L}{R}\frac{d^2y}{dt^2} + \frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{dx(t)}{dt} \tag{5}$$

Substituting $L = 0$, $R =$5, and $C = 1/20$ here yields

$$\frac{dy(t)}{dt} + 4y(t) = \frac{dx(t)}{dt} = 2\cos(2t) \tag{6}$$

Zero-input response of the system: Using the procedure outlined in Appendix C in textbook, and last two lectures we determine the characteristic equation for Eq. (6) as

$(s + 4) = 0$. So, root is $s = -4$ and he zero-input response is

$$y_{zi}(t) = Ae^{-4t} \tag{7}$$

The value of A is obtained from the initial condition $y(0^-) = 2V$. Substituting $y(0^-) = 2V$ in the above equation yields $A = 2$. The zero-input response is given by $y_{zi}(t) = 2e^{-4t}$.

LTIC Example

Zero-state response of the system:

The zero-state response is calculated by solving Eq. (6) with a zero initial condition, $y(0^-) = 0$. The homogeneous component of the zero-state response of Eq. (6) is similar to the zero input response and is given by

$$y_{zi}(t) = Ce^{-4t} \tag{8}$$

where C is a constant. The particular component of the zero-state response of Eq. (6) for input $x(t) = sin(2t)$ is of the following form:

$$y_{zs}^{(p)} = K_1 \cos(2t) + K_2 \sin(2t) \tag{9}$$

Substituting the particular component in Eq.(6) gives $K_1 = 0.4$ and $K_2 = 0.2$. The overall zero-state response of the system is as follows:

$$y_{zs} = Ce^{-4t} + 0.4 \cos(2t) + 0.2 \sin(2t) \tag{10}$$

with zero initial condition, i.e. $y_{zs} = 0$.

LTIC Example

Substituting the initial condition in the zero-state response yields $C = -0.4$. The total response of the system is the sum of the zero-input and zero-state responses and is given by

$$y(t) = 1.6e^{-4t} + 0.4\cos(2t) + 0.2\sin(2t) \tag{11}$$

LTIC Example

Steady state value of the output:

Let see what happens, when $t \to \infty$

$$\begin{aligned}\lim_{t\to\infty} y(t) &= \lim_{t\to\infty}[1.6e^{-4t} + 0.4\cos(2t) + 0.2\sin(2t)] \\ &= 0.4\cos(2t) + 0.2\sin(2t) \\ &= \sqrt{0.4^2 + 0.2^2}\sin(2t + 63.4^\circ)\end{aligned} \tag{12}$$

We can verify that we can get steady state solution using circuit theory (see textbook p. 109)

Solution of 1st order ODE

The output of a first-order differential equation,

$$\frac{dy}{dt} + f(t)y(t) = r(t) \tag{13}$$

resulting from input $r(t)$ is given by

$$y(t) = e^{-p}\left[\int e^{p} r dt + C\right] \tag{14}$$

where function p is given by

$$p(t) = \int f(t)dt \tag{15}$$

and C is a constant determined from initial condition. In our example $f(t) = 4$. We can verify, that using formulas above we we get same answer. See textbook p. 108

LTIC Example 3.3

Lets do another example (Example 3.3 in the textbook) with same RLC circuit as in previous example

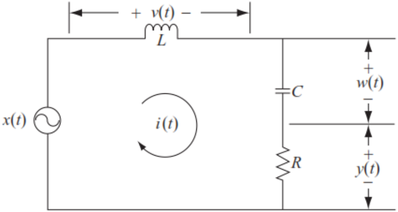

Figure 2: RLC circuit

Consider the electrical circuit shown in Fig.2 with the values of inductance, resistance, and capacitance set to $L = 1/12H, R = 7/12\Omega$, and$C = 1F$. The circuit is assumed to be open before $t = 0$, i.e. no current is initially flowing through the circuit. However, the capacitor has an initial charge of 5 V.

LTIC Example 3.3

Determine:

(i) the zero-input response $w_{zi}(t)$ of the system;

(ii) the zero-state response $w_{zs}(t)$ of the system; and

(iii) the overall output $w(t)$

when the input signal is given by $x(t) = 2exp(-t)u(t)$ and the output $w(t)$ is measured across capacitor C.

Solution LTIC Example 3.3

We got DE for $w(t)$ (See lecture notes 11 eq.(19) subsection "x(t) and w(t)")

$$LC\frac{d^2w}{dt^2} + RC\frac{dw(t)}{dt} + w(t) = x(t) \tag{16}$$

Substituting here L, R, and C above yields

$$\frac{d^2w}{dt^2} + 7\frac{dw}{dt} + 12w(t) = 12x(t), \tag{17}$$

and we have initial conditions, $w(0^-) = 5$ and $\dot{w}(0^-) = 0$, and the input signal is given by $x(t) = 2e^{-t}u(t)$.

Zero-input response of the system: We can find that characteristic equation of eq (17) is $s^2 + 7s + 12 = 0.$ which has roots at $s = -4, -3.$ The zero-input response is therefore given by

$$w_{zi}(t) = (Ae^{-4t} + Be^{-3t})u(t) \tag{18}$$

The value of A and B is obtained from the initial condition $w(0^-) = 5V$ and $\dot{w}(0^-) = 0.$

Solution LTIC Example 3.3

We get following equations.

$$\begin{aligned} A + B &= 5, \\ 4A + 3B &= 0 \end{aligned} \tag{19}$$

which have the solution $A = -15$ and $B = 20$. The zero-input response is therefore given by

$$w_{zi}(t) = (20e^{-3t} - 15e^{-4t})u(t) \tag{20}$$

Zero-state response of the system:

To calculate the zero-state response of the system, the initial conditions are assumed to be zero, i.e. the capacitor is assumed to be uncharged. Hence, the zero-state response $w_{zs}(t)$ can be calculated by solving the following differential equation:

$$\frac{d^2w}{dt^2} + 7\frac{dw}{dt} + 12w(t) = 12x(t), \tag{21}$$

with initial conditions, $w(0^-) = 0$ and $\dot{w}(0^-) = 0$, and input $x(t) = 2exp(-t)u(t)$.

Solution LTIC Example 3.3

The homogeneous solution of Eq.(21) above has the same form as the zero-input response and is given by

$$w_{zs}^{(h)}(t) = C_1 e^{-4t} + C_2 e^{-3t} \tag{22}$$

where C_1 and C_2 are constants.
The particular solution for input $x(t) = 2e^{-t}u(t)$
is of the form $w_{zs}^{(p)}(t) = Ke^{-t}u(t)$. Substituting the particular solution into Eq.(21) and solving the resulting equation yields $K = 4$. The zero-state response of the system is, therefore, given by

$$w_{zs} = (C_1 e^{-4t} + C_2 e^{-3t} + 4e^{-t})u(t). \tag{23}$$

Last step is to compute C_1 and C_2. To do this we need to use initial conditions for zero-state $w(0^-) = 0$ and $\dot{w}(0^-) = 0$. Substituting in this conditions eq.(23) we get

Solution LTIC Example 3.3

$$\begin{aligned} C_1 + C_2 + 4 &= 0, \\ -4C_1 - 3C_2 - 4 &= 0, \end{aligned} \tag{24}$$

Solutions are $C_1 = 8$ and $C_2 = -12$. The zero-state solution is, therefore, given by

$$w_{zs} = (8e^{-4t} - 12e^{-3t} + 4e^{-t})u(t). \tag{25}$$

Overall response of the system The overall response of the system can be obtained by summing up the zero-input and zero-state responses, and can be expressed as

$$w(t) = (-7^{-4t} + 8e^{-3t} + 4e^{-t})u(t). \tag{26}$$

Solution LTIC Example 3.3

w_{zi}, w_{zs} and w are plotted on Fig.3

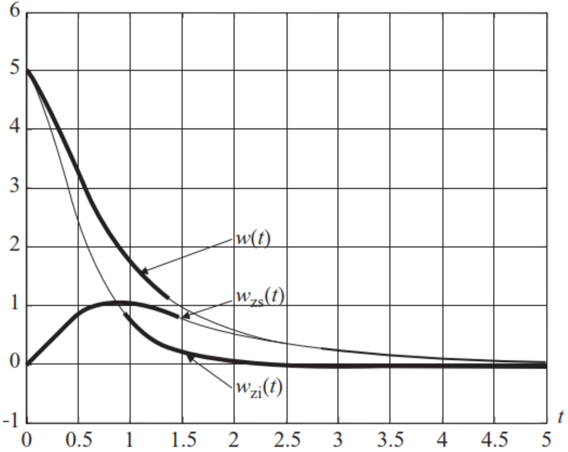

Figure 3: w_{zi}, w_{zs} and w

Representation of signals using Dirac delta functions

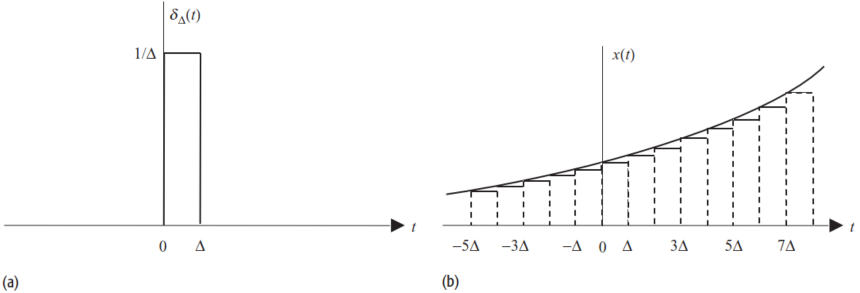

Fig. 3.3. Approximation of a CT signal $x(t)$ by a linear combination of time-shifted unit impulse functions. (a) Rectangular function $\delta_\Delta(t)$ used to approximate $x(t)$. (b) CT signal $x(t)$ and its approximation $\hat{x}(t)$ shown with the staircase function.

We now show that any signal can be represented as a linear combination of time-shifted impulse(δ) functions.

Representation of signals using Dirac delta functions

To illustrate our result, we define a new function $\delta_\Delta(t)$ as follows:

$$\delta_\Delta(t) = \begin{cases} 1/\Delta & 0 < t < \Delta \\ 0 & \text{otherwise} \end{cases} \tag{27}$$

Then any function $x(t)$ can be approximated as

$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t - k\Delta) \cdot \Delta \tag{28}$$

Now if we go to limit $\Delta \to 0$, then $\hat{x}(t) \to x(t)$, which gives

$$x(t) = \lim_{\Delta\to 0} \hat{x}(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t - k\Delta) \cdot [(k+1)\Delta - k\Delta] \tag{29}$$

Representation of signals using Dirac delta functions

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \qquad (30)$$

This result shows, that a CT function can be represented as a weighted superposition of time-shifted impulse functions. We will use this equation to calculate the output of an LTIC system. (There is alternate proof of equation (30) in textbook p. 113)